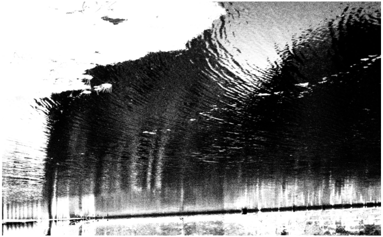

Quantum Propagators and Sheaves

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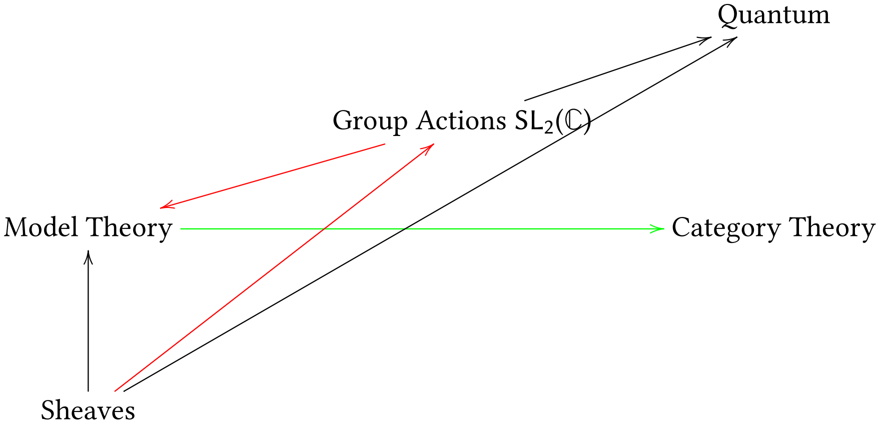
Contents

Some interactions



Plan

Ideal (limit) models in Physics - Goals

Various questions (classical and recently posed or revisited) in Physics point towards the need of various kinds of "ideal structures", and tools of contrast between "real structures" and those ideal (limit) structures.

I plan to illustrate three of these questions, examine some of their answers and contrast with the tools of generic models. I will also provide some side questions for discussion, coming from more model-theoretic considerations.

Zilber: Structural approximation

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huge progress achieved by physicists in dealing with singularities and non-convergent sums and integrals (...Feynman path integrals) has not been matched so far ... with an adequate mathematical theory (Zilber, [Zil])

Implicit knowledge by the physicist of the structure of his model, not yet available to mathematicians? (Rabin, Rieffel, Zeidler)

Limit and ideal models, à la Zilber

Quoting Zilber's [Zil]:

The process of understanding the physical reality by working in an ideal model can be interpreted as follows. We assume that the ideal model $\mathbb{M}_{\text{ideal}}$ is being chosen from a class of "nice" structures, which allows a good theory. We suppose that the real structure $\mathbb{M}_{\text{real}}$ is "very similar" to $\mathbb{M}_{\text{ideal}}$ (...) approximated by a sequence $\mathbb{M}_i$ of structures and $\mathbb{M}_{\text{real}}$ is one of these, $\mathbb{M}_i = \mathbb{M}_{\text{real}}$ sufficiently close to $\mathbb{M}_{\text{ideal}}$. The notion of approximation must also contain both logical and topological ingredients. (...)

This goes on...

... the reason that we wouldn't distinguish two points in the ideal model $\mathbb{M}_{\mathsf{ideal}}$ is that the corresponding points are very close in the real world $\mathbb{M}_{\mathsf{real}}$ so that we do not see the difference (using the tools available). In the limit of the $\mathbb{M}_{\mathsf{i}}$'s this sort of difference will manifest itself as an infinitesimal. In other words, the limit passage from the sequence $\mathbb{M}_{\mathsf{i}}$ to the ideal model $\mathbb{M}_{\mathsf{ideal}}$ must happen by killing the infinitesimal differences. (...) This corresponds to taking a specialization (...) from an ultraproduct $\prod_{\mathsf{D}} \mathbb{M}_{\mathsf{i}}$ to $\mathbb{M}_{\mathsf{ideal}}$.

His examples of structural approximation include no less than the Gromov-Hausdorff limit of metric spaces and deformation of algebraic varieties.

BUT...

... We note that the scheme is quite delicate regarding metric issues. In principle we may have a well-defined metric (...) on the ideal structure only. Existence of a metric, especially the one that gives rise to a structure of a differentiable manifold, is one of the key reasons of why we regard some structures as “nice” or “tame”. The problem of whether and when a metric on $\mathbb{M}$ can be passed to approximating structures $\mathbb{M}_i$ might be difficult, indeed we don’t know how to answer this problem in some interesting cases.

Zilber's approach to structures for Physics

In a nutshell... Zariski Geometries:

$$\mathbb{M} = (M, \mathcal{C})$$

where M is a set and $\mathcal{C}$ is a collection of basic predicates. $\mathcal{C}$ is a basis of closed sets for a topology on each M^n such that

- Projections are $pr : M^n \to M^k$ are continuous.
- Closed sets "are linear, surfaces"... there is a dimension $\dim R$ of every closed set such that if R is irreducible

$$\dim R = \dim pr(R) + \dim(\text{gen.fiber})$$

- (Presmoothness) U irred. is presmooth if for every irred. rel. closed subsets $S_1, S_2 \subset U$ and any irreducible component S_0 of $S_1 \cap S_2$

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim U.$$

Hrushovski-Zilber's theorem

Theorem (Classification Theorem - Hrushovski-Zilber)

Any one-dimensional Zariski geometry $\mathbb{M}$ *that is "non-linear" is associated to a smooth algebraic curve* C *over an algebraically closed field* F *through a surjective map* $p : M \to C(F)$*, definable in* $\mathbb{M}$ *in such a way that the fibres are all of some finite size* N*.*

So, Zariski geometry is "almost" algebraic geometry, but the structure of the finite fibers has been studied by Zilber and found to contain "jewels" of information.

There are "not enough" definable coordinate functions $M \to F$ to encode all the structure of $\mathbb{M}$ - the usual coordinate algebra gives just $C(\mathbb{F})$. In [Zil2]

Zilber's Structural Approximation

Given a topological structure $\mathbb{M}$ and a family of structures $\mathbb{M}_i$, $i \in I$, in the same language, $\mathbb{M}$ is approximated by $\mathbb{M}_i$ along an ultrafilter D on I if for some elementary extension $M^D \succ \prod \mathbb{M}_i/D$ of the ultraproduct there is a surjective homomorphism

$$\lim_D : \mathbb{M}^D \to \mathbb{M}.$$

Examples

These include:

1. The Gromov-Hausdorff limit of metric spaces along a non-principal ultrafilter D.
2. Structural approximation of a quantum torus at q by quantum tori at roots of unity.

Kochen-Specker's Impossibility Theorem

The common sense belief that "every physical quantity must have a value even if we do not know what it is" is challenged in Quantum Physics at the level of the formalism itself: Kochen and Specker proved in 1967 the impossibility of assigning values to all physical quantities while preserving the functional relations between them. This has a sheaf "model theoretical" flavor that was first noticed by Domenech, Freytes and De Ronde, who built a first sheaf theoretic analysis of the theorem.

Döring and Isham have constructed a sheaf "spectral presheaf" that, within the topos-theoretic realm, captures Kochen-Specker as the non-existence of global sections for those spectral presheaves, when the Hilbert space has dimension ≥ 2.

Abramsky, Brandenburger - a framework for non-locality.

- Fix a set X of measurements and a set O of possible outcomes for each measurement.

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$$\mathcal{D}\mathcal{E}(U') \to \mathcal{D}\mathcal{E}(U) :: d \mapsto d \restriction U$$
- The existence of a **global section** for such a sheaf ("empirical model") implies the existence of a local deterministic hidden-variable model.

Plan

Limits

Theorem (A classical Generic Model Theorem)

Let $\mathbb{F}$ *be a generic filter for a sheaf of topological structures* $\mathfrak{A}$ *over* X*. Then*

$$\begin{aligned}\mathfrak{A}[\mathbb{F}] \models \varphi(\sigma/\sim_{\mathbb{F}}) \quad &\iff \quad \{\mathsf{x} \in \mathsf{X} | \mathfrak{A} \Vdash_{\mathsf{x}} \varphi^{\mathrm{G}}(\sigma(\mathsf{x}))\} \in \mathbb{F} \\ &\iff \quad \exists \mathsf{U} \in \mathbb{F} \text{ such that } \mathfrak{A} \Vdash_{\mathsf{U}} \varphi^{\mathrm{G}}(\sigma).\end{aligned}$$

Here, φ^{G} is a formula equivalent classically to φ, but not necessarily in an intuitionistic framework! (The formula φ^{G} is sometimes called the Gödel translation of φ - in 1925, Kolmogorov had independently defined an equivalent translation.)

More on the Generic Model Theorem

Cohen's construction of generic models for set theory is the first published result along these lines. Later, Robinson, Barwise and Keisler used generic model theorems to get Omitting Types Theorems in various logics, generalized by Caicedo. Ellerman's "ultrastalk theorem" (1976) is a GMTh for maximal filters. Miraglia also proves a similar result for Heyting-valued models.

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$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

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$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

Then, the GMTh just means that in the new sheaf $\mathbb{A}^\infty$ this fiber is classic:

$$\mathbb{A}^\infty \Vdash_\infty \varphi(\sigma_1^*, \cdots, \sigma_n^*) \Leftrightarrow \mathbb{A}[\mathbb{F}] \models \varphi([\sigma_1^*], \cdots, [\sigma_n^*])$$

Sheaves of Hilbert Spaces

Why?

1. Hilbert Spaces are (still) a crucial tool for formalization of concepts and objects in Physics and in Chemistry
2. In Physics: really algebras of operators acting on Hilbert spaces.
3. In Chemistry: really predicates on Hilbert spaces.
4. In both, the dynamical properties of evolution of a system are relevant.

In the case of Chemistry, the current treatment is unsatisfactory: capturing the relevant predicates (chemical structure, chemical reaction) has depended on physics to a degree that some theoretical chemists consider excessive.

The problem of a model theory for Hilbert Spaces

So, we want to be able to put Hilbert spaces (and more structure on top of them, such as predicates for reactions, or operators for observables) on fibers.
We could in principle do that as we have seen so far, but immediately we get the problem that we may get lots of non-standard Hilbert spaces (infinitesimals, etc.).
Moreover, we want the logic to "keep track" of (say) the distance to a projection $p(v)$, the convergence of a sequence in H, isometric isomorphism, $(1+\varepsilon)$-isomorphism, etc. etc.
Finally, we need to be able to take limits of Cauchy sequences at will in our structures: metric completeness is crucial.

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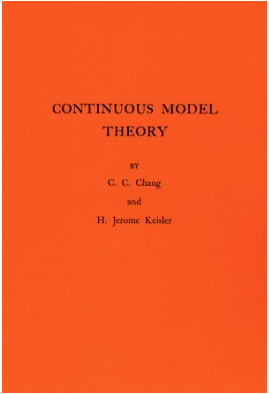
Finally, we need to be able to take limits of Cauchy sequences at will in our structures: metric completeness is crucial.
That is the rôle of Continuous Model Theory.

Continuous Model Theory - Origins



Although the origins of CMTh go way back (von Neumann, Chang & Keisler (1966), and in some (restricted) ways to von Neumann's Continuous Geometryrecent takes on Continuous Model Theory are based on formulations due to Ben Yaacov, Usvyatsov and Berenstein of Henson and Iovino's Logic for Banach Spaces.

Continuous predicates and functions

Definition

Fix (M, d) a bounded metric space. A continuous n-ary predicate is a uniformly continuous function

$$P : M^n \to [0, 1].$$

A continuous n-ary function is a uniformly continuous function

$$f : M^n \to M.$$

Metric structures

Therefore, metric structures are of the form

$$\mathcal{M} = \left(M, d, (f_i)_{i\in I}, (R_j)_{j\in J}, (a_k)_{k\in K}\right)$$

Each function, relation must be endowed with a modulus of uniform continuity.

Metric structures

Therefore, metric structures are of the form

$$\mathcal{M} = \left(M, d, (f_i)_{i \in I}, (R_j)_{j \in J}, (a_k)_{k \in K} \right)$$

where the R_i and the f_j are (uniformly) continuous functions with values in $[0, 1]$, the a_k are distinguished elements of M.
Remember: M is a bounded metric space.
Each function, relation must be endowed with a modulus of uniform continuity.

Examples of FO metric structures

Example

- Any FO structure, endowed with the discrete metric.

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- For a probability space $(\Omega, \mathcal{B}, \mu)$, construct a metric structure $\mathcal{M}$ based on the usual measure algebra of $(\Omega, \mathcal{B}, \mu)$.
- Representations of C^*-algebras (Argoty, Berenstein, Ben Yaacov, V.).
- Valued fields.

The syntax

1. Terms: as usual.
2. Atomic formulas: $d(t_1, t_n)$ and $R(t_1, \cdots, t_n)$, if the t_i are terms. Formulas are then interpreted as functions into $[0, 1]$.
3. Connectives: continuous functions from $[0, 1]^n \to [0, 1]$. Therefore, applying connectives to formulas gives new formulas.
4. Quantifiers: $\sup_x \varphi(x)$ (universal) and $\inf_x \varphi(x)$ (existential).

Interpretation

The logical distance between $\varphi(x)$ and $\psi(x)$ is $\sup_{a \in M} |\varphi^M(a) - \psi^M(a)|$.
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Notice also that the set of connectives is too large, but it may be "densely" and uniformly generated by $0, 1, x/2, \dot{-}$: for every ε, for every connective $f(t_1, \cdots, t_n)$ there exists a connective $g(t_1, \cdots, t_n)$ generated by these four by composition such that $|f(\vec{t}) - g(\vec{t})| < \varepsilon$.

"Continuous Model Theory" beyond First Order

Several contexts, some unexplored so far.

1. Metric Abstract Elementary Classes (Hirvonen, Hyttinen - ω-stability, V. Zambrano - superstability, domination, notions of independence): an amalgam of the power of Abstract Elementary Classes with metric ideas.

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2. Continuous $L_{\omega_1\omega}$. So far, no published results as such. There are however “Lindström theorems” for Continuous First Order due to Caicedo and Iovino.
3. Sheaves of (metric) structures. Our work with Ochoa, motivated by problems originally in Chemistry. NEXT!

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Sheaves of Metric Structures

A sheaf of metric structures $\mathfrak{A}$ over X consists of:

1. A sheaf (E, p) over X,
2. On every fiber $p^{-1}(x)$ $(x \in X)$, a metric structure

$$(\mathfrak{A}_x, d_x) = (E_x, (R_i^x)_i, (f_j^x)_j, (c_k^x)_k, d_x, [0, 1])$$

such that $E_x = p^{-1}(x)$, (E_x, d_x) is a complete bounded metric space of diameter 1, and

- For every i, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^x$ is open
- For every j, $f_j^{\mathfrak{A}} = \bigcup_{x \in X} f_j^x$ is continuous
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(further requirements on moduli of uniform continuity)

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- The premetric $d^{\mathbb{A}} := \bigcup_{x \in X} d_x : \bigcup_{x \in X} E_x^2 \to [0,1]$ is a continuous function.

(further requirements on moduli of uniform continuity)

Truth Continuity - adapted to metric

Truth Continuity is still the guiding paradigm. Remember in the "discrete" case, negation was the first stumbling block - the first place where forcing was needed in a non-trivial way.
Here, in "CFO" logic, the semantics is defined on conditions of the form

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Negation in continuous, metric logic, is weak: the semantics really treats $\leq$ and $\geq$ as "negations" of each other...

Truth continuity happens without the need of forcing in two basic cases:

- Formulas φ composed of max, min, $\dot{-}$ and inf: $\mathbb{A}_x \models \varphi(x) < \varepsilon$ if and only if this happens at all y near x
- Similarly for $\varphi > \varepsilon$ when φ is built of max, min, $\dot{-}$ and sup.

Pointwise forcing

With Ochoa, we define $\mathbb{A} \Vdash_x \varphi < \varepsilon$ and $\mathbb{A} \Vdash_x \varphi > \varepsilon$, for $x \in X$:

- Atomic: $\mathbb{A} \Vdash_x d(t_1, t_2) < \varepsilon \Leftrightarrow d_x(t_1^{\mathbb{A}_x}, t_2^{\mathbb{A}_x}) < \varepsilon$
  $\mathbb{A} \Vdash_x d(t_1, t_2) > \varepsilon \Leftrightarrow d_x(t_1^{\mathbb{A}_x}, t_2^{\mathbb{A}_x}) > \varepsilon$
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- $\mathbb{A} \Vdash_x \max(\varphi, \psi) < \varepsilon \Leftrightarrow \mathbb{A} \Vdash_x \varphi < \varepsilon$ and $\mathbb{A} \Vdash_x \psi < \varepsilon$. Sim. for $>$.
- $\mathbb{A} \Vdash_x \min(\varphi, \psi) \Leftrightarrow \mathbb{A} \Vdash_x \varphi$ or $\mathbb{A} \Vdash_x \psi$. Sim. for $>$.
- $\ldots$

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- $\mathbb{A} \Vdash_x \min(\varphi, \psi) \Leftrightarrow \mathbb{A} \Vdash_x \varphi$ or $\mathbb{A} \Vdash_x \psi$. Sim. for $>$.
- $\mathbb{A} \Vdash_x 1 \dot{-} \varphi < \varepsilon \Leftrightarrow \mathbb{A} \Vdash_x \varphi > 1 \dot{-} \varepsilon$. Sim. for $>$.
- $\mathbb{A} \Vdash_x \varphi \dot{-} \psi < \varepsilon$ iff and only if one of the following holds:
  - $\mathbb{A} \Vdash_x \varphi < \psi$
  - $\mathbb{A} \nVdash_x \varphi < \psi$ and $\mathbb{A} \nVdash_x \varphi > \psi$
  - $\mathbb{A} \Vdash_x \varphi > \psi$ and $\mathbb{A} \Vdash_x \varphi < \psi + \epsilon$.
- $\mathbb{A} \Vdash_x \varphi \dot{-} \psi > \varepsilon$ iff $\mathbb{A} \Vdash_x \varphi > \psi + \varepsilon$
- $\ldots$

Pointwise forcing - continued

Quantifiers:

- $\mathbb{A} \Vdash_x \inf_{s \in A_x} \varphi(s) < \varepsilon$ iff there exists a section σ such that $\mathbb{A} \Vdash_x \varphi(\sigma) < \varepsilon$.

Pointwise forcing - continued

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- $\mathbb{A} \Vdash_x \inf_s \varphi(s) > \varepsilon$ iff there exists an open set $U \ni x$ and a real number $\delta_x > 0$ such that for every $y \in U$ and every section σ defined on y, $\mathbb{A} \Vdash_y \varphi(\sigma) > \varepsilon + \delta_x$

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- $\mathbb{A} \Vdash_x \sup_s \varphi(s) < \varepsilon$ iff there exists an open set $U \ni x$ and a real number $\delta_x > 0$ such that for every $y \in U$ and every section σ defined on y, $\mathbb{A} \Vdash_y \varphi(\sigma) < \varepsilon - \delta_x$
- $\mathbb{A} \Vdash_x \inf_{s \in A_x} \varphi(s) > \varepsilon$ iff there exists a section σ such that $\mathbb{A} \Vdash_x \varphi(\sigma) > \varepsilon$.

Around truth continuity

- We have $\mathbb{A} \Vdash_x \inf_s(1 \dot{-} \varphi) > 1 \dot{-} \varepsilon$ if and only if $\mathbb{A} \Vdash_x \sup_s \varphi < \varepsilon$.

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- We can also define $\mathbb{A} \Vdash_x \varphi(s) \leq \varepsilon$ iff $\mathbb{A} \nVdash_x \varphi(s) > \varepsilon$ and dually for $\geq$.
- With this, for $0 < \varepsilon' < \varepsilon$, if $\mathbb{A} \Vdash_x \varphi(s) \leq \varepsilon'$ then $\mathbb{A} \Vdash_x \varphi(s) < \varepsilon'$

A metric on sections? (Not yet)

So far so good, but we have (for the time being) lost the metric on the sections (so, the corresponding presheaves $\mathbb{A}(U)$ are still missing the “metric” feature - they do not live in the correct category yet).

- Sections have different domains
- Triangle inequality is tricky
- Restrict to sections with domains in a filter of open sets
- But the ultralimit (even in that case) could fail to be complete!

Rather... a pseudometric

Fix F a filter of open sets of X. For all sections σ and μ with domain in F define

$$\mathsf{F}_{\sigma\mu} = \{\mathsf{U} \cap \mathrm{dom}(\sigma) \cap \mathrm{dom}(\mu) | \mathsf{U} \in \mathsf{F}\}.$$

Then the function

$$\rho_{\mathsf{F}}(\sigma, \mu) = \inf_{\mathsf{U} \in \mathsf{F}_{\sigma\mu}} \sup_{\mathsf{x} \in \mathsf{U}} \mathsf{d}_{\mathsf{x}}(\sigma(\mathsf{x}), \mu(\mathsf{x}))$$

is a pseudometric on the set of sections with domain in F.

Completeness of the induced metric

Theorem (Ochoa)

Let $\mathbb{A}$ *be a sheaf of metric structures defined over a regular topological space* X*. Let* F *be an ultrafilter of regular open sets. Then, the metric induced by* ρ_{F} *on* $\mathbb{A}[\mathsf{F}]$ *is complete.*

Local Forcing for Metric Structures

Forcing over an open set is somewhat more tricky in this case. We have the following definition.

Definition

Let $\mathbb{A}$ be a sheaf of metric structures defined on X, $\varepsilon > 0$, U open in X, $\sigma_1, \cdots, \sigma_n$ sections defined on U. Then

- $\mathbb{A} \Vdash_U \varphi(\sigma) < \varepsilon \iff \exists\delta < \varepsilon \forall x \in U(\mathbb{A} \Vdash_x \varphi(\sigma) < \delta)$
- $\mathbb{A} \Vdash_U \varphi(\sigma) > \delta \iff \exists\varepsilon > \delta \forall x \in U(\mathbb{A} \Vdash_x \varphi(\sigma))$

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- $\mathbb{A} \Vdash_{\mathsf{U}} \varphi(\sigma) < \varepsilon \iff \exists\delta < \varepsilon \forall \mathsf{x} \in \mathsf{U}(\mathbb{A} \Vdash_{\mathsf{x}} \varphi(\sigma) < \delta)$
- $\mathbb{A} \Vdash_{\mathsf{U}} \varphi(\sigma) > \delta \iff \exists\varepsilon > \delta \forall \mathsf{x} \in \mathsf{U}(\mathbb{A} \Vdash_{\mathsf{x}} \varphi(\sigma))$

There is an involved, equivalent, inductive definition. We also have $\mathbb{A} \Vdash_{\mathsf{U}} \inf_{\sigma}(1 \dot{-} \varphi(\sigma) > 1 \dot{-} \varepsilon \iff \mathbb{A} \Vdash_{\mathsf{U}} \sup_{\mathsf{U}} \varphi(\sigma) < \varepsilon$, and a maximal principal principle (existence of witnesses of sections).

Metric Generic Model and the Theorem

For the appropriate notion of genericity, we build the generic model as in the discrete case. The definition of genericity guarantees the completeness of $\mathbb{A}[\mathsf{F}]$.

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Theorem (Metric GMTh)

Let F *be a generic filter on* X*,* $\mathbb{A}$ *a sheaf of metric structures on* X *and* $\sigma_1, \cdots, \sigma_n$ *sections. Then*

1. $\mathbb{A}[\mathsf{F}] \models \varphi([\sigma_1]/\sim_{\mathsf{F}}, \cdots, [\sigma_n]/\sim_{\mathsf{F}}) < \varepsilon \iff \exists \mathsf{U} \in \mathsf{F}$ *such that* $\mathbb{A} \Vdash_{\mathsf{U}} \varphi(\sigma_1, \cdots, \sigma_n) < \varepsilon$
2. $\mathbb{A}[\mathsf{F}] \models \varphi([\sigma_1]/\sim_{\mathsf{F}}, \cdots, [\sigma_n]/\sim_{\mathsf{F}}) > \varepsilon \iff \exists \mathsf{U} \in \mathsf{F}$ *such that* $\mathbb{A} \Vdash_{\mathsf{U}} \varphi(\sigma_1, \cdots, \sigma_n) > \varepsilon$

Plan

A Metric Sheaf for noncommuting observables with continuous spectra

Really, a metric sheaf space for a free particle:

Definition

The triple $\mathfrak{A}_{\mathsf{cont}} = (\mathsf{E}, \mathsf{X}, \pi)$ where

- $\mathsf{X} = \mathbb{R}^+$ is the base space with the product topology.
- For $\tau \in \mathsf{X}$ we let E_τ be a two sorted metric model where
  - $\mathcal{U}_\tau$ and $\mathcal{V}_\tau$ span the universe for each sort.
  - Every sort has is a metric space with the metric induced by the norm in $\mathcal{L}^2(\mathbb{R})$.
  - Every sort is a model in the language of a vector space, with symbols for the binary transformation $\langle , \rangle_{\mathcal{V}}$ and $\langle , \rangle_{\mathcal{U}}$, to be interpreted such that

A Metric Sheaf for noncommuting observables with continuous spectra

- $$\langle q(x_0-x)\phi_{(\tau,t_1)}(x_0-x), r(x_1-x)\phi_{(\tau,t_1)}(x_1-x)\rangle_{\mathcal{U}} = q(x_0-x_1)r(x_0-x_1)\phi_{(\tau,t_1+t_2)}(x_0-x_1) \quad (1)$$
  $$\langle q(p_0-p)\phi_{1/(\tau,t_1)}(p_0-p), r(p_1-p)\phi_{1/(\tau,t_1)}(p_1-p)\rangle_{\mathcal{V}} = q(p_0-p_1)r(p_0-p_1)\phi_{1/(\tau,t_1+t_2)}(p_0-p_1) \quad (2)$$
- function symbols for FT and FT^{-1} to be interpreted as in Eq. (??).
- The sheaf is constructed as the disjoint union of fibers: $E = \sqcup_{\tau\in X} E_\tau$
- Sections are defined such that if $\tau \in U \subset X$,
  $$\sigma_{q,x_0,p_0,t}(\tau) = \left(q(x-x_0\phi_{(\tau,t)}(x,x_0)\,,\; q(p-p_0)\phi_{1/(\tau,t)}(p,p_0)\right).$$
- π, the local homeomorphism, is given by $\pi(\psi) = \tau$ if $\psi \in E_\tau$.

Remarks

- The binary transformations $\langle , \rangle_{\mathcal{U}}$ and $\langle , \rangle_{\mathcal{V}}$ are not the objects usually defined as the inner product in a Hilbert space. Instead, they are our representation for the physical inner product as defined by Dirac in each sort.
- We are interested in two kinds of generic metric models:
    1. In the first kind we look at generic models that capture the limit of vanishing τ, for which we take the nonprincipal ultrafilter induced by the family of open regular sets $\{(0, 1/n) : n \in \mathbb{N}\}$. From the structure of the sheaf defined above, limit elements in the generic model coming from the $\mathcal{U}$ sort with $t = 0$ must approach Dirac's delta in position.
    2. On the other hand, the generic metric model we obtain by taking the nonprincipal ultrafilter induced by the family of open regular sets $\{(n, \infty) : n \in \mathbb{N}\}$ must contain limit elements that represent Dirac's distributions in momentum space.

Whence all this?

Laurent Schwartz's work on distributions[?] helped clarify the notions that Dirac had introduced in the conceptual framework of quantum mechanics. Our sheaf may be understood as a model-theoretic description of the quantum mechanics of the position and momentum operators in a subset of the Schwartz space.

Definition

The Schwartz space on $\mathbb{R}^n$ is the function space given by

$$S(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty \quad \forall \alpha, \beta \in \mathbb{N}^n \right\}$$

where α, β are "multi-indices", and $C^\infty(\mathbb{R}^n)$ is the set of smooth complex valued functions from $\mathbb{R}^n$, and

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right|.$$

Whence all this?

Recall the initial issue in quantum mechanics:
The axiomatic framework of quantum mechanics dictates that every *observable* must be described by a self-adjoint operator acting in an appropriate Hilbert Space. Thus, we expect to find operators for position and momentum, $\hat{x}$ and $\hat{p}$ respectively, with domain in the Hilbert space of the system, representing such observables[**?**]. In Dirac notation, the theory claims the existence of elements in the Hilbert space, denoted by $|x\rangle$ and $|p\rangle$, such that the eigenvalue equations $\hat{x}|x\rangle = x|x\rangle$ and $\hat{p}|p\rangle = p|p\rangle$ hold, with $x, p \in \mathbb{R}$. For many systems, x and p can take any value in a measurable subset of $\mathbb{R}$ and therefore we call $\hat{x}$ and $\hat{p}$ operators with continuous spectrum.

Whence all this?

The structure (sometimes called “physical Hilbert space”) formed with these operators differs somewhat from the usual definition in Functional Analysis. In particular, the inner product for the physical Hilbert space is not just complex valued, but can take values on the space of distributions. So, how do we make sense of this?

Whence all this?

Using again Dirac's notation, we define the inner product of two position eigenstates $|x_0\rangle$, $|x_1\rangle$ by

$$\langle x_0|x_1\rangle = \delta(x_0 - x_1), \tag{3}$$

where $\delta(x)$ is the Dirac delta function. Likewise,

$$\langle p_0|p_1\rangle = \delta(p_0 - p_1), \tag{4}$$

which implies that neither position nor momentum eigenstates can be normalized (i.e., that their inner product is not a complex number, but a distribution). In addition, the inner product between position and momentum eigenstates is given by

$$\langle p_0|x_0\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ix_0p_0/\hbar}. \tag{5}$$

Whence all this?

The "physical Hilbert" space has two bases: $\{|x\rangle | x \in \mathbb{R}\}$ and $\{|p\rangle | p \in \mathbb{R}\}$, related to one other by

$$|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx\ e^{-ixp/\hbar} |x\rangle, \quad (6)$$

$$|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp\ e^{+ixp/\hbar} |p\rangle, \quad (7)$$

Whence all this?

Letting $\hat{I}$ be the identity operator for the physical Hilbert space and $[A, B] = AB - BA$, we note that $[\hat{x}, \hat{p}] = i\hbar\hat{I}$. This result has as a consequence that the observables $\hat{p}$ and $\hat{x}$ cannot be simultaneously measured in the lab with absolute accuracy (Heisenberg's uncertainty principle). In the basis set of position eigenstates, the representation for position and momentum operators is given by

$$\hat{x} \to M_x, \tag{8}$$

$$\hat{p} \to -i\hbar\frac{\partial}{\partial x}, \tag{9}$$

where M_x is the multiplication operator by the constant x. Thus $\hat{p}$ is a differential operator in $\mathcal{L}^2(\mathbb{R}, \mu)$ and therefore is only defined on a proper subset of the Hilbert Space. We can also find a representations for these operators in the basis given by the momentum eigenstates in which case $\hat{x}$ is a differential operator and $\hat{p}$ a multiplication operator.

Back to the Schwartz Space - and to the Sheaf construction

One motivation for our construction of the sheaf comes from the following definition of Dirac's distribution in $\mathcal{L}^2(\mathbb{R})$:

$$\lim_{\tau\to 0} \frac{1}{\tau\sqrt{\pi}} e^{-x^2/\tau^2} = \delta(x) \tag{10}$$

(with the limit taken in the sense of distributions). This suggests that an *imperfect*[1] representation $\phi_\tau(x, x_0)$ for the physical vector state $|x_0\rangle$ in $\mathcal{L}^2(\mathbb{R})$ is

$$\phi_\tau(x, x_0) = \frac{1}{\tau\sqrt{2\pi\hbar}} e^{-(x-x_0)^2/2\hbar\tau^2}. \tag{11}$$

The family of elements $\{\phi_\tau(x, x_0)\}$ is a subset of the Schwartz space and, with the inner product in $\mathcal{L}^2(\mathbb{R})$, we find that

$$\langle \phi_\tau(x, x_0), \phi_\tau(x, x_1)\rangle = \int_{-\infty}^{\infty} dx \phi_\tau(x, x_0)\phi_\tau(x, x_1) = \phi_{\sqrt{2}\tau}(x_1, x_0). \tag{12}$$

[1] In the sense of "up to τ"

Back to the Schwartz Space - and to the Sheaf construction

Next, we show how this metric sheaf enables us to do the computation of the quantum mechanical amplitude for a free particle. The energy eigenstates of a physical systems in quantum mechanics are characterized by the Hamiltonian operator $\hat{H}$ and, in the case of a free particle this corresponds to

$$\hat{H} = \frac{\hat{p}^2}{2m}. \tag{13}$$

Back to the Schwartz Space - and to the Sheaf construction

In terms of this operator, we define the quantum mechanical propagator $K(x_1, x_0, t)$ for a free particle as it “travels” from x_0 to x_1 in configuration space by

$$K(x_1, x_0, t) = \langle x_1, U(t)x_0 \rangle \tag{14}$$

with

$$U(t) = e^{-it\hat{H}/\hbar} = e^{-it\hat{p}^2/2m\hbar}. \tag{15}$$

This propagator represents the probability amplitude for the event of a particle traveling from x_0 to x_1 and it has been studied by Zilber and Hirvonen-Hyttinen from a model-theoretic perspective.

Back to the Schwartz Space - and to the Sheaf construction

After a somewhat lengthy calculation, we get

$$\begin{aligned} \langle x_1, U(t)x_0\rangle &= \langle \phi_\tau(x, x_1), \phi_{(\tau, it/m)}(x, x_0)\rangle_{\mathcal{U}} && (16) \\ &= \phi_{(\tau, it/m)}(x_1, x_0) && (17) \\ &= \frac{1}{\sqrt{2\pi(\tau^2 + it/m)}} e^{-(x_1 - x_0)^2/2\hbar(\tau^2 + it/m)} && (18) \end{aligned}$$

Back to the Schwartz Space - and to the Sheaf construction

The previous is the *imperfect* propagator at the fiber E_τ. If we were to take the limit $\tau \to 0$ in this expression we will recover the exact form for the quantum mechanical amplitude, and this is precisely what our choice of the ultrafilter in the base space does: We take the nonprincipal ultrafilter induced by the family of open regular sets $\{(0, 1/\mathsf{n}) : \mathsf{n} \in \mathbb{N}\}$. Thus in the Generic model $\mathfrak{A}[\mathbb{F}]$ we recover the exact propagator as a limit element.

Lógica de los haces de estructuras
Xavier Caicedo
Revista de la Academia Colombiana de Ciencias Exactas, Físicas y Naturales, XIX, no. 74, (1995) 569-585

"What is a thing?": Topos Theory in the Foundations of Physics
Andreas Döring and Chris Isham
in New Structures of Physics, ed. R. Coecke, Springer, 2008.

Sheaves of metric structures.
Maicol Ochoa and Andrés Villaveces
Logic, Language, Information, and Computation. 23rd International Workshop, WoLLIC 2016, Puebla, Mexico, August 16-19th, 2016. Proceedings
Editors: Väänänen, Jouko, Hirvonen, Åsa, de Queiroz, Ruy (Eds.)

Springer Berlin. 2016 - pp. 297-315.

Quantum Mechanics in a metric sheaf: a model theoretic approach.
Maicol Ochoa and Andrés Villaveces
Preprint, submitted
arXiv:1702.08642

On model theory, non-commutative geometry and physics.
Boris Zilber
Bulletin of Symbolic Logic, 2010.

Non-commutative Zariski geometries and their classical limit.
Boris Zilber
arXiv0900.4415

¡Gracias!